<u>Unit 4 – Graphs of Trigonometric Functions - Classwork</u>

For each of the angles below, calculate the values of $\sin x$ and $\cos x$ (2 decimal places) on the chart and graph the points on the graph below.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°	
$y = \sin x$																		
$y = \cos x$																		
	<u> </u>																	
↑ 1	-			_										_	_			_
1	y																	
															-		_	
	-			_											_	_		→
b		50			100		1	50		200		25	50		300		35	0
T												2	c					
															_			
1																		
T																		

What you are seeing are the graphs of the sine and cosine functions. Since the function values repeat, you are looking at one cycle of the curves. We call these curves **periodic** because of their repetitive nature.

Let us look at these curves on the calculator. For right now, let's stay in degree mode.



Let's note the similarities and differences of the sine and cosine curves:

Similarities

Differences

Since the curves are similar in shape, we call them **sinusoids**.

Note that since these are periodic functions, we can look at as many periods as we wish:



Also notice that we can graph both the sine and cosine function in radian mode:







Xmin= Xmax≕

scl

max

min≡



the window was set up with Xmax as 2π and Xscl as $\frac{\pi}{2}$

ZoomTrig (Degree) ZoomTrig (Radian)

5

While we will be exploring most of our graph at first in degree mode, please realize that graphs scaled correctly should be graphed in radian mode because you are comparing numbers on the x-scale to numbers on the y-scale. Graphing in Degree mode is comparing degrees on the x-scale to numbers on the y-scale.

When we graph a sinusoid within its primary period of $[0,2\pi)$ or $[0^\circ,360^\circ)$, there are 5 points that help us in sketching the curve. We call them the **critical points** of the sinusoid. By definition, the critical points are the points in which the curve either crosses it axis of symmetry, reached a high point (maximum) or low point (minimum).

			Radians or Deg	rees	
	0 or 0°	$\frac{\pi}{2}$ or 90°	π or 180°	$\frac{3\pi}{2}$ or 270°	2π or 360°
$y = \sin x$	$(0,0)$ or $(0^\circ,0)$	$\left(\frac{\pi}{2},1\right)$ or $\left(90^\circ,1\right)$	$(\pi, 0)$ or $(180^{\circ}, 0)$	$\left(\frac{3\pi}{2},-1\right) \text{ or } (270^\circ,-1)$	$(2\pi, 0)$ or $(360^{\circ}, 0)$
$y = \cos x$	$(0,1)$ or $(0^{\circ},1)$	$\left(\frac{\pi}{2},0\right)$ or $\left(90^\circ,0\right)$	$(\pi, -1)$ or $(180^\circ, -1)$	$\left(\frac{3\pi}{2},0\right)$ or $\left(270^\circ,0\right)$	$(2\pi, 1)$ or $(360^{\circ}, 1)$



Amplitude and Period

Now that we know the shape and behavior of the sine and cosines curves, we will now do some things to alter the behavior. Our goal is to predict the shape of the curve without resorting to actually graphing it.

On your calculators, graph the curves $y = \sin x$, $y = 2\sin x$, $y = 4\sin x$, $y = -.5\sin x$, $y = -3\sin x$ in degree mode on the window below.



Notice that by changing the coefficient of the function, we control its scaling factor -a vertical stretch or vertical shrink of the basic sine curve. We call this the amplitude of the curve - the height of the curve above its axis of symmetry.

The **amplitude** of $y = a \sin x$ or $y = a \cos x$ is the largest value of y and is given by |a|. The range of the curve is [-a, a]. We define the shape of the curve using this chart: $\begin{cases} a > 0 \dots$ curve normal $a < 0 \dots$ curve reversed $a < 0 \dots$ curve reversed $a < 0 \dots$ curve reversed $a = 1 \dots$ vertically stretch amplitude $a < 1 \dots$ vertically shrunk amplitude $a = 1 \dots$ overtical change

Example 1) For each of the curves below, find the amplitude, range, and 5 critical points (using degrees). Then graph it on the calculator using an appropriate window to show the behavior of the graph.

	Curve	Amplitude	Range	Critical Points	Shape (circle)	Vertical (circle)
a.	$y = 3\cos x$				Normal / Reverse	Stretch / Shrink
b.	$y = -2\sin x$				Normal / Reverse	Stretch / Shrink
c.	$y = \cos x$				Normal / Reverse	Stretch / Shrink
d.	$y = 7\sin x$				Normal / Reverse	Stretch / Shrink
e.	2 .				Normal / Reverse	Stretch / Shrink
	$y = -\frac{1}{3}\sin x$					
f.	$y = -\pi \cos x$				Normal / Reverse	Stretch / Shrink

On your calculators, graph the curves $y = \sin x$, $y = \sin 2x$, $y = \sin 4x$, $y = \sin 5x$ in degree mode on the window on the next page.



Notice that by changing the coefficient of the *x*-variable, we control the number of cycles of the curve we produce. In $y = \sin x$, when the coefficient of *x* is 1, we get one cycle within 360°. In $y = \sin 2x$, we get two cycles within 360°. Thus we also change the **period** of the curve, the number of degrees (or radians) the curve takes to complete one cycle.

The **period** of $y = \sin bx$ or $y = \cos bx$ is the number of degrees (or radians) the curve takes to complete one cycle. It is found using this formula: Period = $\begin{cases} \frac{360^{\circ}}{b} & \text{for degrees} \\ \frac{2\pi}{b} & \text{for radians} \end{cases}$ We define the horizontal stretch in words: $\begin{cases} \text{period} < 360^{\circ} & (\text{or } 2\pi) & \dots & \text{compressed} \\ \text{period} > 360^{\circ} & (\text{or } 2\pi) & \dots & \text{elongated} \\ \text{period} = 360^{\circ} & (\text{or } 2\pi) & \dots & \text{none} \end{cases}$ To find the 5 critical points, we use 0, $\frac{\text{period}}{4}$, $\frac{\text{period}}{2}$, $\frac{3\text{period}}{4}$, period whether we are in degrees or radians.

Example 2) For each of the curves below, find the period (use degrees), horizontal stretch, and 5 critical points. Then graph it on the calculator using an appropriate window to show the behavior of the graph.

	Curve	Period	Horizontal Stretch (circle)	Critical Points
a.	$y = \cos 2x$		Compressed / elongated	
b.	$y = \sin 3x$		Compressed / elongated	
c.	$y = \cos 4x$		Compressed / elongated	
d.	$y = \sin 0.5x$		Compressed / elongated	
e.	$y = \cos\frac{2}{3}x$		Compressed / elongated	
f.	$y = \sin\frac{3}{2}x$		Compressed / elongated	
g.	$y = \cos 10x$		Compressed / elongated	

Example 3) Now lets put both amplitude and period together. For each of the curves below, find all the pertinent information.

	Curve	Amplitude	Period	Critical Points	Shape	Vertical	Horizontal
					(circle)	Stretch (circle)	Stretch (circle)
•	$v = 2 \sin r$				Normal	Stretched	Compressed
a.	$y = 2 \sin x$				Reversed	Shrunk	Elongated
h	$v = \sin 2r$				Normal	Stretched	Compressed
U.	$y = \sin 2x$				Reversed	Shrunk	Elongated
0	$v = 2 \sin 2r$				Normal	Stretched	Compressed
C.	$y = 2 \sin 2x$				Reversed	Shrunk	Elongated
4	$v = 4\cos^2 r$				Normal	Stretched	Compressed
a.	y = 40053x				Reversed	Shrunk	Elongated
0	$v = -3\cos 6r$				Normal	Stretched	Compressed
e.	y = -50050x				Reversed	Shrunk	Elongated
	1.1				Normal	Stretched	Compressed
f.	$y = -\frac{\sin -x}{2}$				Reversed	Shrunk	Elongated
	. 4				Normal	Stretched	Compressed
g.	$y = \sin \frac{1}{3}x$				Reversed	Shrunk	Elongated
1	3.3				Normal	Stretched	Compressed
h.	$y = -\frac{1}{4}\sin\frac{1}{4}x$				Reversed	Shrunk	Elongated

Translations of Sinusoids

On your calculators, graph $y = \sin x$, $y = 2 + \sin x$, $y = -1 + \sin x$.



Note by changing the constant that is added or subtracted to the basic sine or cosine curve, we affect how the graph of the sinusoid is shifted up or down. This is called **vertical translation** or **vertical shift**.

A curve in the form of $y = d \pm a \sin bx$ or $y = d \pm a \cos bx$ will shift the curves $y = \pm a \sin bx$ or $y = \pm a \cos bx$ up or down based on the value of d. This is called the vertical translation of the curve. $\begin{cases}
If <math>d > 0...vertical \text{ translation is } up d \text{ units} \\
If <math>d < 0...vertical \text{ translation is } down d \text{ units} \\
If <math>d = 0...\text{there is no vertical translation}
\end{cases}$

Finally, on your calculators, graph the functions $y = \sin x$, $y = \sin(x - 30^\circ)$, $y = \sin(x + 45^\circ)$



Note by changing the constant that is added or subtracted within the parentheses, we shift the sinusoid left or right. This is called **horizontal translation** or **phase shift**.

Critical Points = 0, $\frac{\text{period}}{4}$, $\frac{\text{period}}{2}$, $\frac{3\text{period}}{4}$, period True Critical Points = c, $\frac{\text{period}}{4} + c$, $\frac{\text{period}}{2} + c$, $\frac{3\text{period}}{4} + c$, period + c

Example 4) For each curve, find the vertical translation, phase shift, Critical Points and True Critical Points.

	Curve	Vertical Translation	Phase shift / direction	Critical Points True Critical Points
a.	$y = 4 + \sin x$			
b.	$y = \cos(x - 60^\circ)$			
c.	$y = 5 + \sin(x + 20^\circ)$			
d.	$y = -2 - \cos\left(x + \frac{\pi}{6}\right)$			
e.	$y = \sin\left(x - \frac{\pi}{4}\right) - 1$			

Let's put it all together. Given the curves $y = d \pm a \sin b(x - c)$ or $y = d \pm a \cos b(x - c)$ The **amplitude** = |a|We define the shape of the curve using this chart: $\begin{cases} a > 0... \text{curve normal} \\ a < 0... \text{curve reversed} \end{cases}$ We define vertical change using this chart: $\begin{cases} \text{amplitude} > 1... \text{vertically stretch} \\ \text{amplitude} < 1... \text{vertically shrunk} \end{cases}$ The **period** = $\frac{360^{\circ}}{b}$ (degrees) or $\frac{2\pi}{b}$ (radians) We define the horizontal stretch in words: $\begin{cases} \text{period} < 360^{\circ} \text{ (or } 2\pi) \dots \text{ compressed} \\ \text{period} > 360^{\circ} \text{ (or } 2\pi) \dots \text{ elongated} \end{cases}$ The **phase shift** (or horizontal translation) = c. If c > 0, the curve shifts right, if c < 0, the curve shifts left

If c > 0, the curve shifts right, if c < 0, the curve shifts left Critical Points = $0, \frac{\text{period}}{4}, \frac{\text{period}}{2}, \frac{3\text{period}}{4}, \text{period}$ True Critical Points = $c, \frac{\text{period}}{4} + c, \frac{\text{period}}{2} + c, \frac{3\text{period}}{4} + c, \text{period} + c$

The Vertical Translation = d

 $\begin{cases} If d > 0...vertical translation is up d units \\ If d < 0...vertical translation is down d units \end{cases}$

The range of the function is
$$\left[d - a, d + a\right]$$

Example 5) For each curve, find all pertinent information and verify it on your calculator. (The way you verify it is to input the function, and graph it. Your xMin and xMax should be the smallest and largest of the true critical points. Your yMin and yMax should be the smallest and largest values of the range. If your answer is correct, you will see exactly one period of the curve.

	Curve:	Amp-	Pd.	Phase Shift	Critical Pts	Vert	Range	Shape	Vertical	Horizontal
		litude		& Direction	True Crit. Pts	Tran		(circle)	Stretch	Stretch
									(circle)	(circle)
2	$v = 3 - \cos r$						r 1	Normal	Stretched	Compressed
а.	$y = 5 \cos x$						L ']	Reversed	Shrunk	Elongated
h	$v = 1 + 2\cos 2r$						۲ I	Normal	Stretched	Compressed
0.	y = 1 + 20032x						L ']	Reversed	Shrunk	Elongated
0	$v = -2 = 3\sin 4r$						r 1	Normal	Stretched	Compressed
υ.	$y = -2 - 5 \sin 4x$						L ']	Reversed	Shrunk	Elongated
d	$v = 5\sin(r - 30^\circ)$						r 1	Normal	Stretched	Compressed
u.	$y = 5 \sin(x - 50)$						L ']	Reversed	Shrunk	Elongated
•	$v = -1 - \cos(r + 15^{\circ})$						r 1	Normal	Stretched	Compressed
υ.	$y = 1 \cos(x + 15)$						L ']	Reversed	Shrunk	Elongated
c	$2^{1} \cdot 2(5^{\circ})$						r ı	Normal	Stretched	Compressed
I.	$y = 2 + \frac{-\sin 2(x-5)}{2}$						[,]	Reversed	Shrunk	Elongated
	<u> </u>							Normal	Stretched	Compressed
g.	$y = \frac{1}{2} - 4\cos(3x + 15^{\circ})$						[,]	Reversed	Shrunk	Elongated
	$\frac{2}{\sqrt{1-\frac{1}{2}}}$							Keverseu	SIIIUIIK	Eloligated
h	$v = -\frac{3}{3}\sin(\frac{1}{2}r - 40^{\circ})$						۲ I	Normal	Stretched	Compressed
п.	4^{y-4}							Reversed	Shrunk	Elongated
									~	
i.	$y = 5 + 3\sin[2x - \frac{\pi}{2}]$						[,]	Normal	Stretched	Compressed
								Reversed	Shrunk	Elongated

Determining an equation from the graph of a sinusoid



Here is the generalization to determine the equation from its graph: Do the problem above on the right. The equation will be in the form $y = d \pm a \sin b(x - c)$ or $y = d \pm a \cos b(x - c)$

- 1) Decide whether it is a sine or cosine curve. If it "starts" at a high point or low point, it is a cosine curve. If it "starts" in the middle, it is a sine curve. You also must determine if the curve is reversed. If so a < 0.
- 2) Draw the axis of symmetry. That is the value of d.
- 3) Find the height of the curve above the axis of symmetry. That is *a*.
- 4) Find the period by inspection. period = $\frac{360^{\circ}}{b}$ then $b = \frac{360^{\circ}}{\text{period}}$
- 5) Is there a shift? If shifted right, c > 0. If shifted left, c < 0.
- 6) Put it all together.

Example 6) For each of the graphs below, find an equation that generates it. Verify using the calculator.



Finding Trigonometric Models

Suppose you had a trigonometric curve that passed through these points:

x	1	4	7	10	13
у	3	5	3	1	3



Let's develop a method that would generate an equation $y = d \pm a \sin b(x - c)$ or $y = d \pm a \cos b(x - c)$ that fits the data.

First, although it looks like a sine curve, it will generally be easier to consider it a cosine curve that has been shifted 4 units from the *y*-axis. Thus c = 4.

Next, there is an axis of symmetry at the line y = 3. To the right is a drawing of that axis line sketched in. We know that the axis of symmetry is the *d* value, thus d = 3.

	•		
•		• •	-
		•	
<u> </u>			-

We also know the amplitude is the height of the curve above its axis of symmetry, thus a = 2.

Finally the period: We see an entire period of the curve so period = 12. We know that period = $\frac{360^{\circ}}{b}$ or period = $\frac{2\pi}{b}$ but since we are not dealing with degrees, but in radians, we use period = $\frac{2\pi}{b}$ or $b = \frac{2\pi}{\text{period}}$. So $b = \frac{2\pi}{12} = \frac{\pi}{6}$.



There are other possible equations that generate this data. Can you determine others?

Example 8) Find a trigonometric function that passes through these points: Verify on the calculator.

x	3	7	11	15	19
y	5	-1	5	11	5

Harmonic Motion

Many real life applications generate sinusoidal curves. This occurs when data is periodic. Some examples involve electric currents, musical tones, radio waves, sunrises, and weather patterns. We will investigate some:

Example 9) Suppose Tarzan is swinging back and forth on his grapevine. As he swings, he goes back and

forth over the river bank, hovering alternately over land and water. Jane decides to mathematically model his motion and starts her stopwatch.

Let t be the number of second the stopwatch reads and let y be the number of feet Tarzan is from the riverbank (not his height). Assume that y varies sinusoidally with t and that y is positive when Tarzan is over water and negative when Tarzan is over land.

Jane finds that at 2 seconds, Tarzan is at one end of his swing 23 feet from the riverbank over land. At 5 seconds, Tarzan is at the other end of his swing 17 feet from the riverbank over water.



Generate the equation that represents Tarzan's swing model.

Solution: If we can generate 5 points for the sinusoidal motion (harmonic motion), we can use the technique on the previous page to generate the equations.

t			
У			

Once we have our equation, we can answer several questions:

- a) Where was Tarzan when Jane started her stopwatch?
- b) Find the smallest value for *t* when Tarzan was directly over the riverbank.
- c) How long is Tarzan over water?
- d) Where will Tarzan be one minute from the time Jane started the stopwatch?

Example 10) Mark Twain sat on the deck of a river steamboat. As the paddlewheel turned, a brightly painted yellow dot on the paddle blade moved in such a way that its distance from the water's surface was a sinusoidal function of time. When the watch read 4 seconds, the dot was at its highest point, 16 feet above the water's surface. The wheel's diameter was 18 feet and it completed a revolution every 10 seconds.



Generate an equation for the sinusoidal curve and answer these questions:

- a) How far above the surface was the dot when Mark's stopwatch read 5 seconds?
- b) How long was the dot below the water's surface?
- c) From the time when the stopwatch was started, when was the first time that the dot was at the surface of the water? At that time was in emerging from or entering the water?

Other Trigonometric Graphs

You are responsible for the graphs of the other trig functions. Note that these trig functions have asymptotes:



Note how the sine and cosecant curves are reciprocals of each other as are the cosine and secant curves. As one goes up, the other goes down and vice versa. At *x*-values where the sine and cosine function is zero, the cosecant and secant functions have vertical asymptotes.

<u>Unit 4 – Graphs of Trigonometric Functions - Classwork</u>

1. For each of the curves below, find all the pertinent information. Then graph on your calculator using an appropriate window.

	Curve	Amplitude	Period	Critical Points	Shape	Vertical	Horizontal
					(circle)	Stretch (circle)	Stretch (circle)
9	$v = 6\cos x$				Normal	Stretched	Compressed
а.	y 0 0 051				Reversed	Shrunk	Elongated
h	$v = -\sin r$				Normal	Stretched	Compressed
υ.	y = 511 x				Reversed	Shrunk	Elongated
	3.				Normal	Stretched	Compressed
C.	$y = -\frac{1}{5}\sin x$				Reversed	Shrunk	Elongated
	3						
d	$v = -\frac{1}{\sin x}$				Normal	Stretched	Compressed
ч.	2 2				Reversed	Shrunk	Elongated
	$v = \cos^3 r$				Normal	Stretched	Compressed
e.	$y = \cos 3x$				Reversed	Shrunk	Elongated
	1				Normal	Stretched	Compressed
f.	$y = \cos - x$				Reversed	Shrunk	Elongated
	3				100001500	Sinunk	Lionguiou
σ	$v = \sin \frac{5}{r}$				Normal	Stretched	Compressed
g.	$y = \sin 3^{x}$				Reversed	Shrunk	Elongated
1					Normal	Stretched	Compressed
n.	$y = 2\cos 3x$				Reversed	Shrunk	Elongated
	1				Normal	Stretched	Compressed
i.	$y = \frac{1}{2}\sin 6x$				Reversed	Shrunk	Flongated
	2				Reversed	Sintunk	Eloligated
i	$v = -4\cos^2 r$				Normal	Stretched	Compressed
J.	$y = -4\cos^{-1}x$				Reversed	Shrunk	Elongated
	1				Normal	Stretched	Compressed
k.	$y = -\sin\frac{1}{10}x$				Reversed	Shrunk	Flongated
	10				iceverseu	Sintunk	Liongatou
1	5_{in}^{2}				Normal	Stretched	Compressed
1.	$y = -\frac{5}{2} \frac{1}{5}x$				Reversed	Shrunk	Elongated

2. For each curve, find all pertinent information and verify it on your calculator.

	Curve:	Amp- litude	Pd.	Phase Shift & Direction	Critical Pts True Crit. Pts	Vert Tran	Range	Shape (circle)	Vertical Stretch (circle)	Horizontal Stretch (circle)
a.	$y = 4 - \frac{1}{2}\cos x$						[,]	Normal Reversed	Stretched Shrunk	Compressed Elongated
b.	$y = 10 + 3\sin 2x$						[,]	Normal Reversed	Stretched Shrunk	Compressed Elongated
c.	$y = 5 + \cos\frac{1}{2}x$						[,]	Normal Reversed	Stretched Shrunk	Compressed Elongated
d.	$y = 1 - 7\cos(x + 30^\circ)$						[,]	Normal Reversed	Stretched Shrunk	Compressed Elongated
e.	$y = \frac{2}{3}\sin 4(x - 5^\circ)$						[,]	Normal Reversed	Stretched Shrunk	Compressed Elongated

	Curve:	Amp- litude	Pd.	Phase Shift & Direction	Critical Pts True Crit. Pts	Vert Tran	Range	Shape (circle)	Vertical Stretch (circle)	Horizontal Stretch (circle)
f.	$y = 2 + 4\cos(3x + 135^\circ)$						[,]	Normal Reversed	Stretched Shrunk	Compressed Elongated
g.	$y = -1 - \frac{4}{5}\sin\frac{3}{2}x$						[,]	Normal Reversed	Stretched Shrunk	Compressed Elongated
h.	$y = 3 + \cos\left(\frac{2}{3}x - 60^\circ\right)$						[,]	Normal Reversed	Stretched Shrunk	Compressed Elongated
i.	$y = \frac{1}{2} + 2\sin\left(3x - \frac{\pi}{2}\right)$						[,]	Normal Reversed	Stretched Shrunk	Compressed Elongated

3. For each of the graphs below, find an equation that generates it. Verify using the calculator



4. Find trigonometric equations that pass through the following points: Verify using the calculator.

a.	x	1.5	1.7	1.9	2.1	2.3
	y	2.1	1.5	.9	1.5	2.1

b.	x	8	18.5	29	39.5	50
	у	-24	-2	20	-2	-24

c.	x	5.27	14.67	24.07	33.47	42.87
	у	6.3	9	-8.1	9	6.3

- 5. A buoy in the ocean is bobbing up and down in harmonic motion. At time = 2 seconds, the buoy is at its high point and returns to that high point every 10 seconds. The buoy moves a distance from 3.6 feet from its high point to its low point. (*hint*: you decide what height represents sea level without the waves and continue accordingly). Determine 5 points and construct an equation that describes this motion.
 - a) How high is the buoy at time = 30 seconds? Is it rising or falling at that time?
 - b) How many times in the 2 minutes will the buoy be at sea level?
- 6. The inside of a bicycle wheel whose diameter is 25 inches is 3 inches off the ground. An ant is sitting on the inside of the wheel. Steve starts riding the bicycle at a steady rate. In 1.2 seconds the ant reaches its highest point on the wheel. The wheel makes a revolution every 1.6 seconds. Determine 5 points and find the equation that describes the motion of the ant.
 - a) What will be the height of the ant 25 seconds into the ride?
 - b) Within the first 10 seconds, how many times will the ant be at its starting height?
- 7. John is bouncing up and down on a trampoline which is 4 feet off the ground. The highest he gets off the trampoline is 11 feet which he reaches in 2 seconds. He completes a bounce every 3 seconds. Determine 5 points and find the equation that describes this motion from the ground.
 - a) In 45 seconds, what will be his height? Is he going up or down at this time?
 - b) Within the first 45 seconds, how many times does he reach his peak?