University of Houston
High School Contest – Spring 2015
Calculus Test

NAME: ______________________________

SCHOOL: ______________________________

1. $\lim\limits_{x \to 0} \tan\left(\dfrac{\sin 2\pi x}{6x}\right) =$

   (a) 0
   (b) $1/\sqrt{3}$
   (c) 1
   (d) $\sqrt{3}$
   (e) The limit does not exist

2. Let f be some function for which you know only that

$$\text{if} \qquad 0 < |x-4| < 1, \qquad \text{then} \qquad |f(x) - 5| < 0.1.$$

   Which of the following statements are necessarily true?

   I. If $|x-4| < 1$, then $|f(x) - 5| < 0.1$.

   II. If $|x-3.5| < 0.3$, then $|f(x) - 5| < 0.1$.

   III. $\lim\limits_{x \to 4} f(x) = 5$.

   IV. If $0 < |x-4| < 0.5$, then $|f(x) - 5| < 0.1$.

   (a) IV only
   (b) II and IV
   (c) I and II
   (d) II and III
   (e) II, III and IV

3. A function f is defined on an interval $[a,b]$. Which of the following statements could be false?

I. If f is differentiable on (a,b) and if f has no zeros on $[a,b]$, then $f(a)$ and $f(b)$ have the same sign.

II. If f is continuous on $[a,b]$, and if $f(a) < 0$ and $f(b) > 0$, then there must be a point $c \in (a,b)$ such that $f(c) = 0$.

III. If f is continuous on $[a,b]$ and there is a point c in (a,b) such that $f(c) = 0$, then $f(a)$ and $f(b)$ have opposite sign.

IV. If f is differentiable on an interval $I \supset [a,b]$, and if $f(a)$ and $f(b)$ have opposite sign, then there must be a point $c \in (a,b)$ such that $f(c) = 0$.

(a) II only
(b) II and IV
(c) I, III and IV
(d) II and III
(e) I and III

4. Set

$$g(x) = \begin{cases} x^3 - 4 & 0 < x < 2, \\ ax^2 + bx & 2 \le x < 5. \end{cases}$$

If g is differentiable on $(0,5)$ then $a + b =$

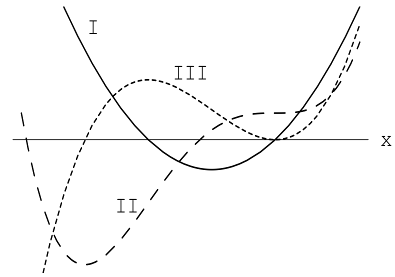
(a) -3
(b) -1
(c) 5
(d) 13
(e) 3

5. If $\dfrac{d}{dx} f(x) = g(x)$ and if $h(x) = e^{2x}$, then $\dfrac{d}{dx} f(h(x)) =$

(a) $g\left(e^{2x}\right)$
(b) $2e^{2x}\, g\left(e^{2x}\right)$
(c) $2e^{2x}\, g(x)$
(d) $e^{2x} g'(x)$
(e) $e^{2x} g\left(e^{2x}\right)$

6. Three graphs labeled I, II and III are shown in the figure. One is the graph of f, one is the graph of f' and one is the graph of f''. Which of the following correctly identifies each of the three graphs?

	f	f'	f''
(a)	I	II	III
(b)	I	III	II
(c)	II	I	III
(d)	II	III	I
(e)	III	II	I



7. Suppose that f is continuous on $[0,4]$ and differentiable on $(0,4)$. Suppose also that $f(0)=5$ and $f(4)=-3$. Which of the following statements is not necessarily true?

(a) The Mean-Value Theorem applies to f.
(b) f is integrable on $[0,4]$.
(c) There exists a number $c \in (0,4)$ such that $f'(c)=-1$.
(d) There exists a number $c \in (0,4)$ such that $f(c)=\pi$.
(e) If c is any number such that $0<c<4$, then $\lim_{x\to c} f(x)$ exists.

8. If $f'(x)=6(x-3)^2(x-9)$, which of the following is true about f?

(a) f has a local maximum at $x=3$ and a local minimum at $x=9$.
(b) f has a point of inflection at $x=3$ and a local maximum at $x=9$.
(c) f has a local minimum at $x=3$ and a local maximum at $x=9$.
(d) f has a point of inflection at $x=3$ and a local minimum at $x=9$.
(e) f has a local maximum at $x=3$ and a point of inflection at $x=9$.

9. An equation for the normal line to the curve $3x^2+2xy+y^2=11$ at the point $(1,2)$ is

(a) $3x-5y+7=0$
(b) $5x+3y=7$
(c) $3x-4y+5=0$
(d) $5x+4y=13$
(e) $4x-5y+6=0$

10. The position of a particle moving along the x-axis is given by

$$s(t) = t^3 - 12t^2 + 45t + 4$$

for $t \geq 0$. For what values of t is the speed of the particle increasing?

(a) $3 < t < 4$ only

(b) $t > 4$ only

(c) $t > 5$ only

(d) $0 < t < 3$ and $t > 5$

(e) $3 < t < 4$ and $t > 5$

11. The curve $x^3 + x \tan y = 27$ passes through $(3, 0)$. Use local linearization to estimate the value of y at $x = 3.1$.

(a) -2.7

(b) -0.9

(c) 0.6

(d) 0.1

(e) -2.1

12. If $\int_0^4 f(x)\,dx = 5$, $\int_2^4 f(x)\,dx = 7$, and $\int_0^7 f(x)\,dx = 10$, then $\int_7^2 f(x)\,dx =$

(a) 12

(b) 8

(c) -12

(d) -2

(e) -8

13. The region in the first quadrant bounded by the graph of $y = e^{-x^2}$, the vertical line $x = 1$, and the x-axis is revolved about the y-axis. Find the volume of the solid that is generated.

(a) π/e

(b) $\pi(e - 1)$

(c) $\pi - \pi/e$

(d) $\pi - e$

(e) $\pi + \pi/e$

14. If f is a continuous function and $F(x) = 4 + \int_0^x \left[t^2 + 3t \int_2^t f(u)\,du\right] dt$, then $F''(2) =$

(a) $10f(2)$
(b) $6f'(2)$
(c) $2 + 6f(2)$
(d) $2 + f(2)$
(e) $4 + 6f(2)$

15. The function $F(x) = 3 + \int_{x^2}^4 \sqrt{4+3t}\,dt$ has an inverse. $\left(F^{-1}\right)'(3) =$

(a) $1/4$
(b) $-1/4$
(c) $1/8$
(d) $-1/16$
(e) $1/16$

16. A container has the shape of an open right circular cone, vertex down. The height of the container is 10 cm and the diameter of the opening (the base of the cone) is 10 cm. Water in the container is evaporating so that its depth is decreasing at the constant rate of 0.3 cm/hr. The rate of change of the volume of water in the container when $h = 5$ cm is

(a) -5.890 cm^3/hr
(b) -1.178 cm^3/hr
(c) -8.562 cm^3/hr
(d) -4.112 cm^3/hr
(e) -6.992 cm^3/hr

17. Let R be the region bounded by the graph of $f(x) = \sqrt{x-1}$, the vertical line $x = 10$, and the x-axis. Find the volume of the solid generated when R is revolved about the line $y = 3$.

(a) $\dfrac{99\pi}{2}$
(b) $\dfrac{135\pi}{2}$
(c) $\dfrac{189\pi}{2}$
(d) $\dfrac{119\pi}{2}$
(e) $\dfrac{137\pi}{2}$

18. A curve in the plane is defined by the parametric equations: $x = 3t^2+2,\ y = \frac{2}{3}\,t^3+1,\ t \in [0,4]$. Find the length of the curve.

    (a) 244/3
    (b) 98
    (c) 196/3
    (d) 250/3
    (e) 196

19. Find a such that $\lim\limits_{x\to 0} \dfrac{e^{ax^2} - \cos 8x}{x^2} = 64$

    (a) 48
    (b) 32
    (c) 28
    (d) 16
    (e) No value of a exists.

20. Evaluate the improper integral $\displaystyle\int_0^\infty \frac{1}{e^x + e^{-x}}\, dx$.

    (a) $\pi/4$
    (b) $\pi/2$
    (c) 1
    (d) $\pi/3$
    (e) divergent

21. Let f be a function such that $|\,f^{(n)}(x)| \le 2$ for all x and n. Find the least integer n such that the Taylor polynomial of degree n at $x = 0$ approximates $f(1/2)$ to within 0.0005.

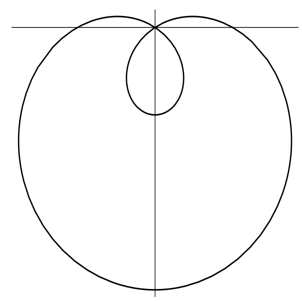
    (a) 2
    (b) 3
    (c) 4
    (d) 5
    (e) 6

22. A 200 gallon tank, initially full of water, develops a leak at the bottom. It is observed that the water is leaking out at a rate proportional to the amount of water present. If 25% of the water leaks out in the first 4 minutes, find the amount of water left in the tank after 10 minutes.

(a) 75.694 gallons
(b) 116.103 gallons
(c) 97.428 gallons
(d) 105.962 gallons
(e) 55.771 gallons

23. The curve $r = 1 - 2\sin\theta$ is shown in the figure. Find the approximate area of the inner loop of the curve.

(a) $\frac{1}{2}\pi - \frac{1}{2}\sqrt{3}$
(b) $\pi - \sqrt{3}$
(c) $2\pi - \sqrt{3}$
(d) $\frac{1}{2}\pi + \sqrt{3}$
(e) $\pi - \frac{3}{2}\sqrt{3}$



24. A curve in the plane has the property that the normal line to the curve at each point $P(x, y)$ always passes through the point $(2, 0)$. Find an equation for the curve given that it passes through the point $(1, 1)$.

(a) $(2 - x) + y^2 = 2$
(b) $(x - 2)^2 + 2y = 3$
(c) $y = e^{x-1}$
(d) $(x - 2)^2 + y^2 = 2$
(e) $\dfrac{(x-2)^2}{3} + \dfrac{2y^2}{3} = 1$

25. The values of x for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2\, 2^n}{x^n}$ converges are

(a) $|x| \geq 1/2$
(b) $|x| > 2$
(c) $-1/2 < x < 1/2$
(d) $-2 \leq x \leq 2$
(e) The series diverges for all x

26. $\{a_n\}$ is a sequence of real numbers. Which of the following statements are necessarily true?

I. If $a_n > 1$ for all n and $a_n \to L$, then $L > 1$.

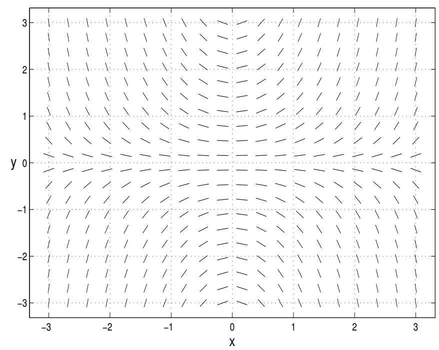
II. If $\{a_n\}$ is not bounded below, then it diverges.

III. If a_n converges, then it is bounded above.

IV. If $\{a_n\}$ is bounded, then it converges.

(a) III only
(b) I, III
(c) II, IV
(d) II, III, IV
(e) II, III

27. Which differential equation has the slope field



(a) $\dfrac{dy}{dx} = xy$
(b) $\dfrac{dy}{dx} = \dfrac{y}{x}$
(c) $\dfrac{dy}{dx} = \dfrac{x}{y}$
(d) $\dfrac{dy}{dx} = x^2 y$
(e) $\frac{dy}{dx} = x + y$

28. Suppose that the power series $\sum_{k=0}^{\infty} a_k (x-1)^k$ converges at $x = 4$. Which of the following series must be convergent?

I. $\sum_{k=0}^{\infty} a_k 4^k$

II. $\sum_{k=0}^{\infty} a_k 2^k$

III. $\sum_{k=0}^{\infty} (-1)^k a_k 3^k$

IV. $\sum_{k=0}^{\infty} (-1)^k a_k$

(a) I only
(b) I and III
(c) III and IV
(d) II and IV
(e) I, II and III

29. $\lim_{n\to\infty} \left(1 - \frac{r}{n}\right)^{2n} =$

(a) 1

(b) e^{-2r}

(c) $-2r$

(d) e^{2r}

(e) e^{-r}

30. A tank with a capacity of 200 gallons is initially full of pure water. At time $t = 0$, salt water with salt concentration 8 ounces/gallon begins to flow into the tank at a rate of 4 gallons/minute. The well-mixed solution in the tank is pumped out at the same rate. Give the total amount of salt in the tank at time $t = 25$ minutes.

(a) 64.23 pounds

(b) 30.75 pounds

(c) 39.35 pounds

(d) 48.62 pounds

(e) 51.91 pounds