OPTIMIZATION

Optimization is the process of finding the greatest or least value of a function for some constraint, which must be true regardless of the solution. In other words, **optimization** finds the most suitable value for a function within a given domain.

(Kaplan Book: Page 380 to 381)

Video: https://www.youtube.com/watch?v=Zq7g1nc2MJ8

https://www.youtube.com/watch?v=EOJbmMB8uCQ

Typical optimization problems ask:

- How large a rectangular area can be enclosed with a fixed amount of fencing?

In this case, the dimensions of the region are non-negative quantities.

- What is the largest volume that can be enclosed by making a box from a rectangular piece of material?

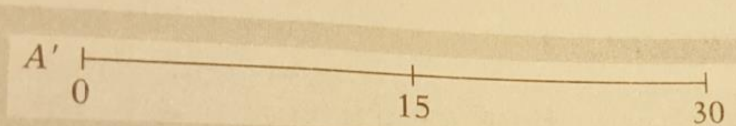
In this case, the dimensions of the box are non-negative quantities.

- What is the minimum distance from a given point to a curve?

In this case, the domain of the distance function is restricted to the domain of the curve given.

The examples that follow illustrate these problems.

Read this after looking at example 1 (Justification of example 1)



Now make a table for values of x and A', choosing values of x from the intervals (0, 15) and (15, 30).

x	5	20
A'	+	−

Since A' changes from + to − at $x = 15$, by the First Derivative Test, $x = 15$ is the x-coordinate of a relative maximum. Since $A(0) = 0$, $A(30) = 0$, and $A(15) > 0$, $A(15) = 225$ is the absolute maximum on [0, 30]. Thus, the length and width that give the maximum area are $l = 15$ ft and $w = 15$ ft.

Optimization

Optimization involves finding the largest or smallest values of a function, such as area, perimeter, volume, length, or time. As with all functions, the domain of the function is vital to determining the solution.

Typical optimization problems ask:

- How large a rectangular area can be enclosed with a fixed amount of fencing?

  In this case, the dimensions of the region are non-negative quantities.

- What is the largest volume that can be enclosed by making a box from a rectangular piece of material?

  In this case, the dimensions of the box are non-negative quantities.

- What is the minimum distance from a given point to a curve?

  In this case, the domain of the distance function is restricted to the domain of the curve given.

The examples that follow illustrate these problems.

EXAMPLE 1

A rectangular vegetable garden is to be enclosed by 60 feet of fencing. Find the length and width that will give the maximum area.

Solution If the length $= x$ and the width $= y$, then the amount of fencing is represented by the equation $2x + 2y = 60$. Solving this equation for y, we have $y = \frac{60 - 2x}{2} = 30 - x$.

Then $A(x) = x(30 - x) = 30x - x^2$.

The dimensions of the rectangle are non-negative quantities. Since $y = 30 - x$ and $y \geq 0$, this means that $x \leq 30$. Thus, the domain of $A(x)$ is $\{0 \leq x \leq 30\}$.

There are two methods of solving for the value of x that will maximize the area function.

METHOD 1 This method uses the fact that the area function is quadratic. Therefore the value of x that will give the maximum value of the function is found by using the formula $x = -\frac{b}{2a}$. (This is also the x-coordinate of the vertex of the graph of the area function.)

Thus $x = \frac{-30}{-2} = 15$, and $y = 30 - x = 15$.

The rectangle that will give the maximum area is a square. This method requires no justification. Since the area function is a parabola that opens down, the vertex must be a maximum.

METHOD 2 Find the critical points of the derivative of the area function $A(x)$ by finding $A'(x) = 30 - 2x$ and solving $A'(x) = 0$. Thus, $x = 15$ and $y = 30 - x = 15$.

This answer must be justified as the absolute maximum by the First Derivative Test and by checking values at the endpoints.

JUSTIFICATION First, make a number line, mark it A', and put the value $x = 15$ on it along with the endpoints of the domain, $x = 0$ and $x = 30$.

EXAMPLE 2

(Variation of Example 1) A rectangular vegetable garden is to be enclosed using the wall of a building as one side and 60 feet of fencing on the other three sides. Find the length and width that will give the maximum area.

Solution If the length = x and the width = y, then the amount of fencing is represented by the equation $2x + y = 60$.

The area of the garden is length × width or $A = xy$.

Solving the first equation for y, we have $y = 60 - 2x$.

The area equation can then be rewritten as

$$A = x(60 - 2x) \text{ or } 60x - 2x^2.$$

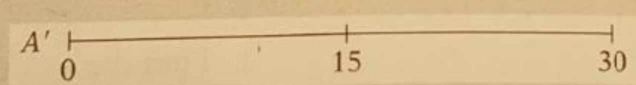
Since x and y are the dimensions of the rectangle, $x \geq 0$ and $y \geq 0$. Since $y = 60 - 2x$ and $y \geq 0$, this means that $x \leq 30$. Thus, the domain of $A(x)$ is $\{0 \leq x \leq 30\}$.

METHOD 1 Use the fact that the area function is quadratic. As described in Method 1, Example 1, $x = \frac{-60}{-4} = 15$ and $y = 60 - 2x = 30$.

METHOD 2 Find the critical points of the derivative of the area function A(x) by finding $A'(x) = 30 - 4x$ and solving $A'(x) = 0$. Thus $x = 15$ and $y = 60 - 2x = 30$.

This answer must be justified as the maximum by the First Derivative Test.

JUSTIFICATION First, make a number line, mark it A', and put the value $x = 15$ on it along with the endpoints of the domain, $x = 0$ and $x = 30$.



Now make a table for values of x and A', choosing values of x from the intervals (0, 15) and (15, 30).

x	5	20
A'	+	−

Since A' changes from + to − at $x = 15$, by the First Derivative Test, $x = 15$ is the x-coordinate of a relative maximum. Since $A(0) = 0$, $A(30) = 0$, and $A(15) > 0$, $A(15) = 450$ is the absolute maximum of $A(x)$ on [0, 30]. Thus, the length and width that give the maximum area are: length = 15 ft, width = 30 ft.

EXAMPLE 3

A box is constructed from a square sheet of cardboard 20 inches on a side by cutting out squares of the same size from each of the four corners and turning up the sides. How long should the side of the square be so that the box has the maximum volume?

Solution If the sides of the squares that are cut out have length x, then the length and width of the box are $20 - 2x$, and the volume of the box is $V(x) = x(20 - 2x)(20 - 2x)$.

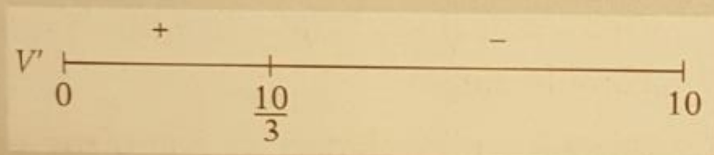
Since the dimensions of the box are all nonnegative, $x \geq 0$ and $20 - 2x \geq 0$. Thus $x \leq 10$, and the domain of $V(x)$ is $\{0 \leq x \leq 10\}$.

$$V = x(20 - 2x)^2$$

$$V' = (20 - 2x)(20 - 6x)$$

Solving $V' = 0$, $x = 10$ or $x = \frac{10}{3}$.

JUSTIFICATION



x	V'
1	+
4	–

The value of V' changes from + to – at $x = \frac{10}{3}$. Thus, $V\left(\frac{10}{3}\right)$ is a relative maximum.

Since $V\left(\frac{10}{3}\right) > 0$, and $V(0) = 0$, and $V(10) = 0$; therefore, $V\left(\frac{10}{3}\right)$ is the absolute maximum of V on $[0, 10]$. The length of the sides of the squares to be cut is $\frac{10}{3}$ inches.

Exercises

Multiple-Choice Questions

A graphing calculator is required for some questions.

1. Two positive numbers have a sum of 10. Find their largest possible product.
   (A) 5
   (B) 10
   (C) 25
   (D) 50
   (E) 100

2. Find the x-coordinate of the point on the graph of $4x + 3y = 7$ that is closest to the origin.
   (A) 0
   (B) 1
   (C) 1.120
   (D) 1.960
   (E) 2.333

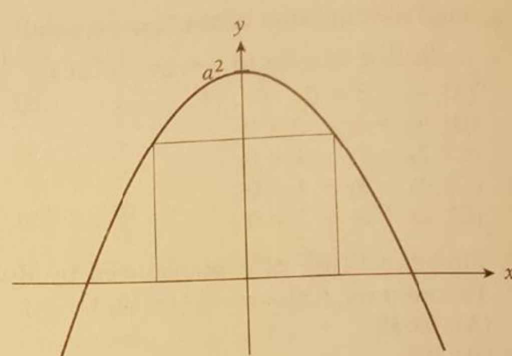
3. A rectangle is inscribed in the semicircle $y = \sqrt{4 - x^2}$. Find its largest possible area.
   (A) 1.4
   (B) $\sqrt{3}$
   (C) $2\sqrt{3}$
   (D) 4
   (E) undefined

4. Squares of equal size are cut off the corners of an 8×10 piece of cardboard. The sides are then turned up to form an open box. What is the largest possible volume of the box?
   (A) 1.472
   (B) 1.5
   (C) 23.986
   (D) 52.50
   (E) 52.514

Free-Response Questions

A graphing calculator is required for some questions.

1. A rectangle is inscribed above the x-axis in the parabola $y = a^2 - x^2$. Find the area of the largest possible rectangle.



2. A rectangular plot is to be fenced in using the side of an existing barn that is 50 feet long as one side of the plot. Two hundred feet of fencing are available for the other three sides of the plot. Find the largest possible area that can be enclosed.

3. Twenty feet of wire are to be used to create a wire sculpture that consists of a square and a circle. Find the largest number of square feet of area that can be enclosed by the square and the circle.

Linearization Videos

1) https://www.youtube.com/watch?v=c33fI_d8GEg

2) https://www.youtube.com/watch?v=BPSNisGXe7U

3) https://www.khanacademy.org/math/differential-calculus/derivative_applications/local-linearization/v/local-linearization-intro

4) https://www.youtube.com/watch?v=Y3jg7HQff_U

13 minutes of physics and linearization.

https://www.youtube.com/watch?v=qBFWbiOJPBA