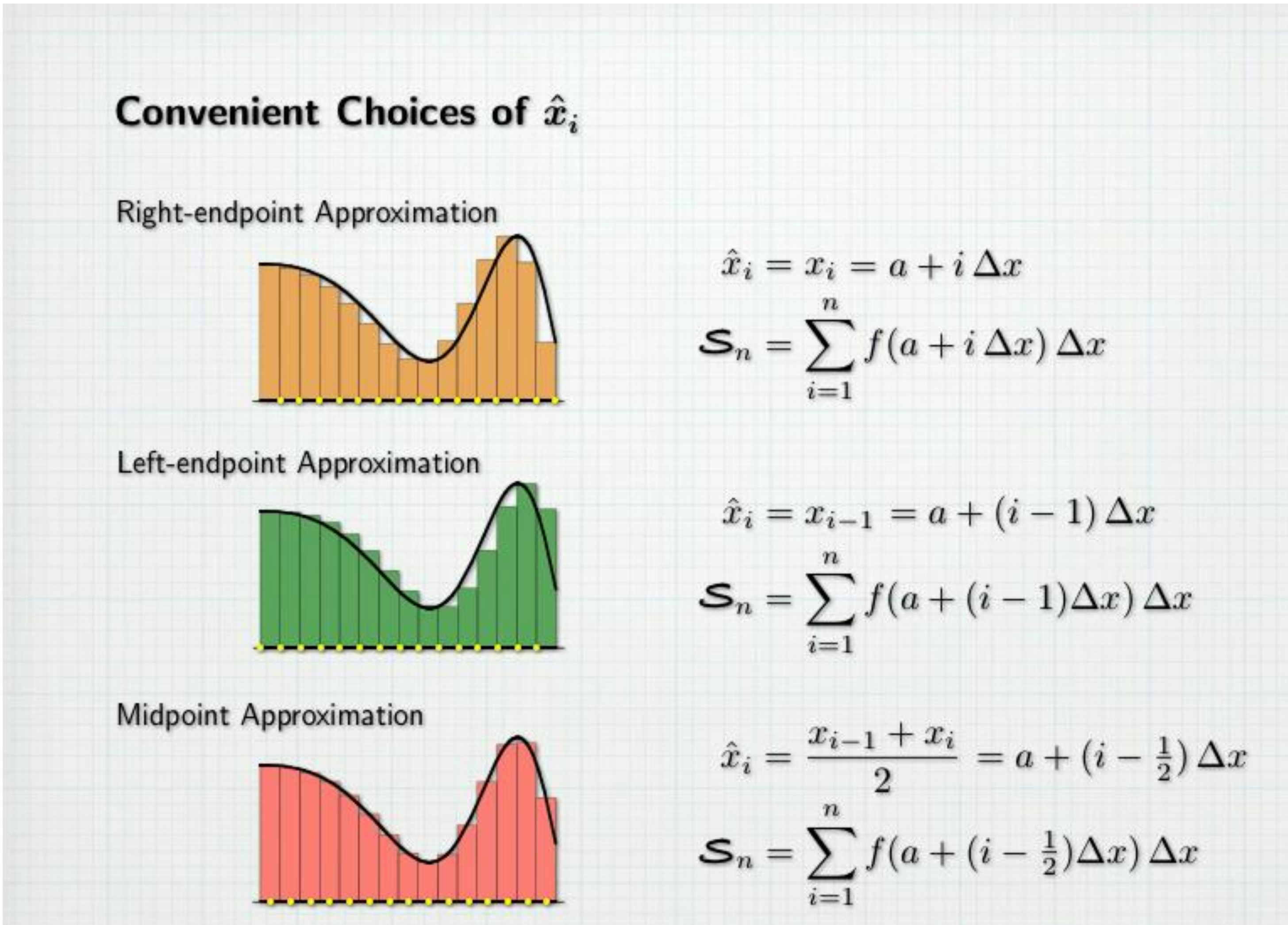
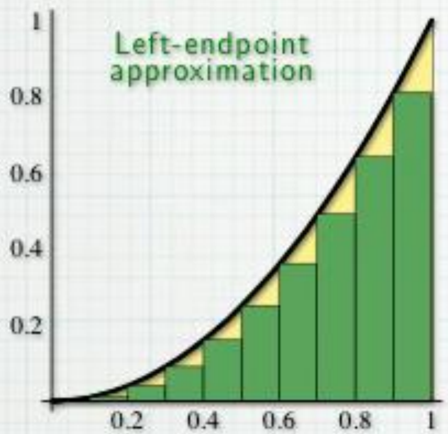
Area under the curve (U.H. video #21)

http://online.math.uh.edu/HoustonACT/videocalculus/SV3/21-area.mov



Example $f(x) = x^2$ on $[0,1]$. $n = 10$ $\Delta x = \frac{1}{10}$

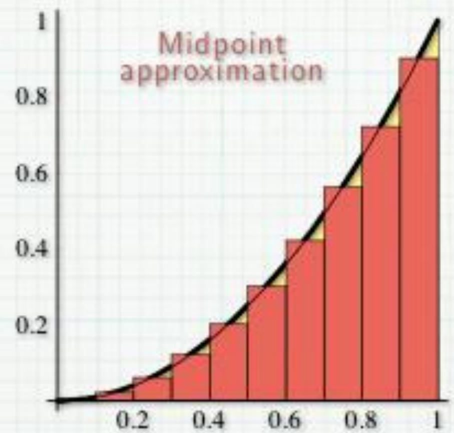


$\hat{x}_i = (i-1)\,\frac{1}{10}$

$$\mathcal{S}_{10} = \sum_{i=1}^{10} \left((i-1)\frac{1}{10}\right)^2 \frac{1}{10}$$

```
sum(seq((.1(I-1)
)²*.1,I,1,10)
                .285
```

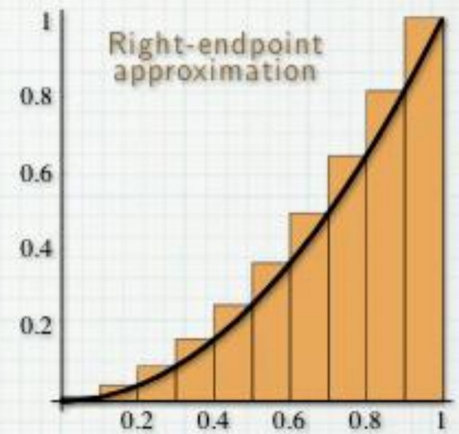
TI-83



$\hat{x}_i = (i-\frac{1}{2})\,\frac{1}{10}$

$$\mathcal{S}_{10} = \sum_{i=1}^{10} \left((i-\tfrac{1}{2})\frac{1}{10}\right)^2 \frac{1}{10}$$

```
sum(seq((.1(I-1)
)²*.1,I,1,10)
                .285
sum(seq((.1(I-.5
))²*.1,I,1,10)
               .3325
```

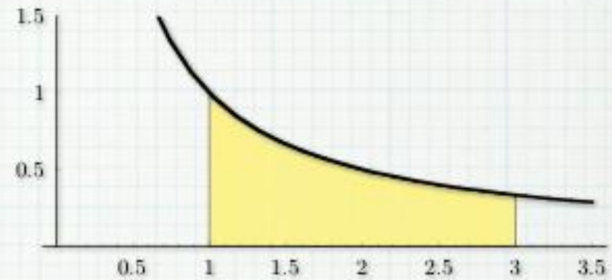


$\hat{x}_i = i\,\frac{1}{10}$

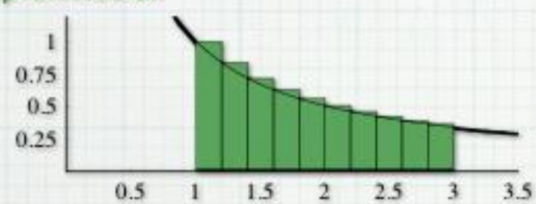
$$\mathcal{S}_{10} = \sum_{i=1}^{10} \left(i\,\frac{1}{10}\right)^2 \frac{1}{10}$$

```
sum(seq((.1(I-1)
)²*.1,I,1,10)
                .285
sum(seq((.1(I-.5
))²*.1,I,1,10)
               .3325
sum(seq((.1*I)²*
.1,I,1,10),10)
```

Example $f(x) = 1/x$ on $[1, 3]$.

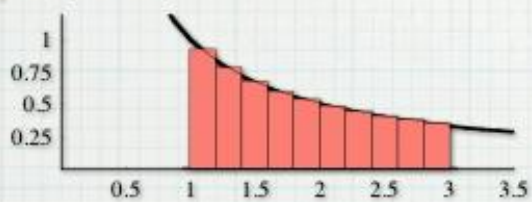


Left-endpoint approximation



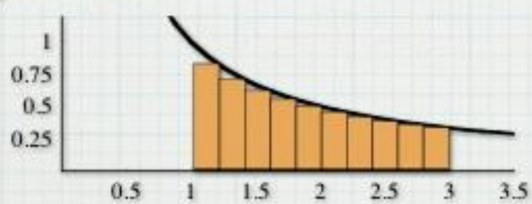
$$\mathcal{S}_{10} = \sum_{i=1}^{10} \frac{1}{1+(i-1)\frac{1}{5}} \frac{1}{5} = \sum_{i=1}^{10} \frac{1}{5+(i-1)} \approx 1.168$$

Midpoint approximation



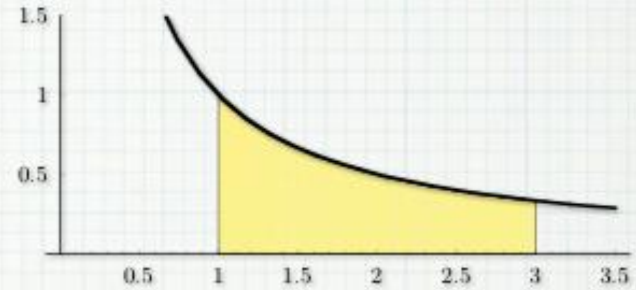
$$\mathcal{S}_{10} = \sum_{i=1}^{10} \frac{1}{1+(i-\frac{1}{2})\frac{1}{5}} \frac{1}{5} = \sum_{i=1}^{10} \frac{1}{5+(i-\frac{1}{2})} \approx 1.097$$

Right-endpoint approximation



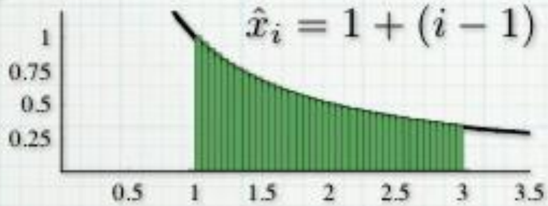
$$\mathcal{S}_{10} = \sum_{i=1}^{10} \frac{1}{1+i\frac{1}{5}} \frac{1}{5} = \sum_{i=1}^{10} \frac{1}{5+i} \approx 1.035$$

Example $f(x) = 1/x$ on $[1,3]$.



$$n = 40 \quad \Delta x = \frac{3-1}{40} = \frac{1}{20}$$

Left-endpoint approximation

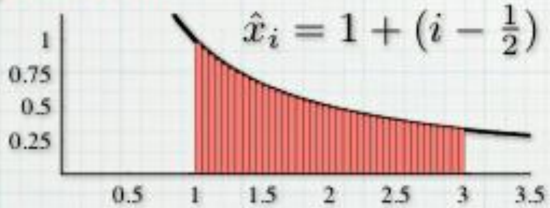


$$\hat{x}_i = 1 + (i-1)\,\frac{1}{20}$$

$$\mathcal{S}_{10} = \sum_{i=1}^{10} \frac{1}{1+(i-1)\frac{1}{5}}\,\frac{1}{5} = \sum_{i=1}^{10} \frac{1}{5+(i-1)} \approx 1.168$$

$$\mathcal{S}_{40} = \sum_{i=1}^{40} \frac{1}{1+(i-1)\frac{1}{20}}\,\frac{1}{20} = \sum_{i=1}^{40} \frac{1}{20+(i-1)} \approx 1.115$$

Midpoint approximation

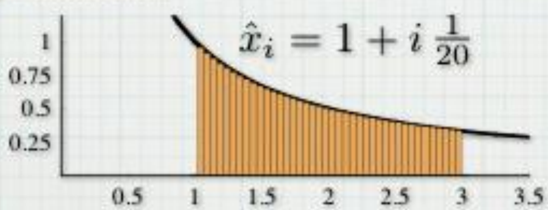


$$\hat{x}_i = 1 + (i-\tfrac{1}{2})\,\frac{1}{20}$$

$$\mathcal{S}_{10} = \sum_{i=1}^{10} \frac{1}{1+(i-\frac{1}{2})\frac{1}{5}}\,\frac{1}{5} = \sum_{i=1}^{10} \frac{1}{5+(i-\frac{1}{2})} \approx 1.097$$

$$\mathcal{S}_{40} = \sum_{i=1}^{40} \frac{1}{1+(i-\frac{1}{2})\frac{1}{20}}\,\frac{1}{20} = \sum_{i=1}^{40} \frac{1}{20+(i-\frac{1}{2})} \approx 1.0985$$

Right-endpoint approximation

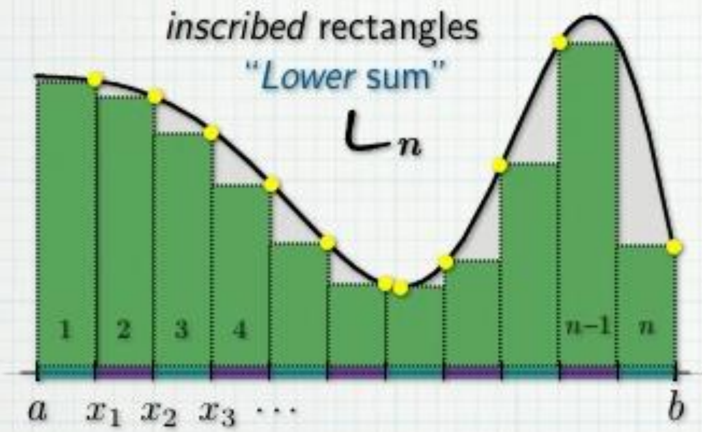


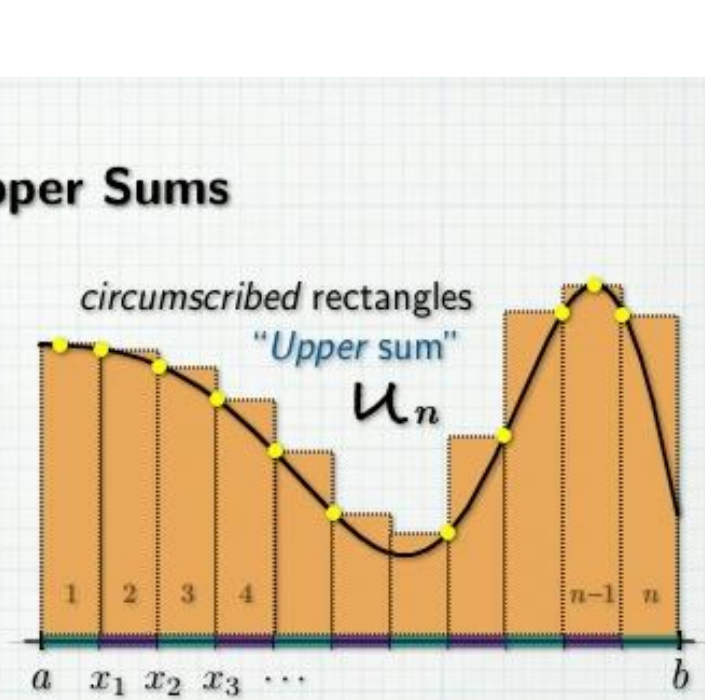
$$\hat{x}_i = 1 + i\,\frac{1}{20}$$

$$\mathcal{S}_{10} = \sum_{i=1}^{10} \frac{1}{1+i\frac{1}{5}}\,\frac{1}{5} = \sum_{i=1}^{10} \frac{1}{5+i} \approx 1.035$$

$$\mathcal{S}_{40} = \sum_{i=1}^{40} \frac{1}{1+i\frac{1}{20}}\,\frac{1}{20} = \sum_{i=1}^{40} \frac{1}{20+i} \approx 1.082$$

Lower and Upper Sums





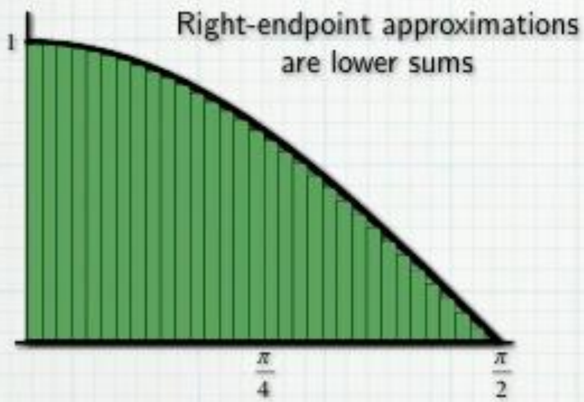
Observations: $\mathcal{L}_m \leq \text{Area} \leq \mathcal{U}_n$ for all m and n

Uniform grid $\mathcal{L}_n \leq \mathcal{L}_{2n} \leq \text{Area} \leq \mathcal{U}_{2n} \leq \mathcal{U}_n$ for all n

$$\mathcal{L}_4 \leq \mathcal{L}_8 \leq \mathcal{L}_{16} \leq \cdots \leq \text{Area} \leq \cdots \leq \mathcal{U}_{16} \leq \mathcal{U}_8 \leq \mathcal{U}_4$$

For each n, $\mathcal{L}_n \leq \mathcal{S}_n \leq \mathcal{U}_n$ for every $\mathcal{S}_n$

Example $f(x) = \cos x$ on $[0, \frac{\pi}{2}]$. $\Delta x = \frac{\pi}{2n}$



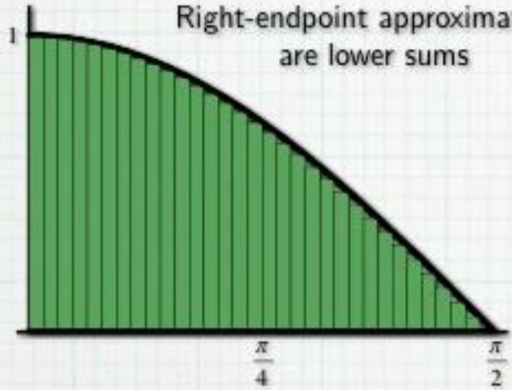
$$L_4 = \sum_{i=1}^{4} \cos\left(i\tfrac{\pi}{8}\right) \tfrac{\pi}{8} \approx 0.7908$$

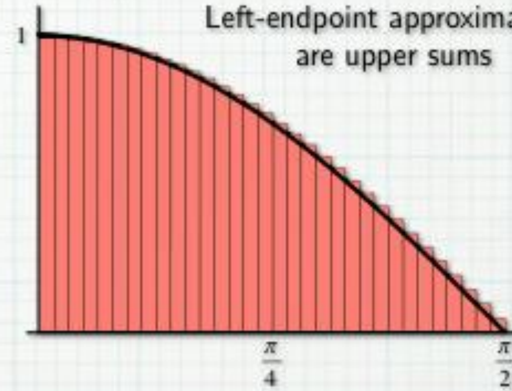
$$L_8 = \sum_{i=1}^{8} \cos\left(i\tfrac{\pi}{16}\right) \tfrac{\pi}{16} \approx 0.8986$$

$$L_{16} = \sum_{i=1}^{16} \cos\left(i\tfrac{\pi}{32}\right) \tfrac{\pi}{32} \approx 0.9501$$

$$L_{32} = \sum_{i=1}^{32} \cos\left(i\tfrac{\pi}{64}\right) \tfrac{\pi}{64} \approx 0.9752$$

Example $f(x) = \cos x$ on $[0, \frac{\pi}{2}]$. $\Delta x = \frac{\pi}{2n}$





$$\mathcal{L}_4 = \sum_{i=1}^{4} \cos\left(i\tfrac{\pi}{8}\right)\tfrac{\pi}{8} \approx 0.7908$$

$$\mathcal{U}_4 = \sum_{i=1}^{4} \cos\left((i-1)\tfrac{\pi}{8}\right)\tfrac{\pi}{8}$$

$$\mathcal{L}_8 = \sum_{i=1}^{8} \cos\left(i\tfrac{\pi}{16}\right)\tfrac{\pi}{16} \approx 0.8986$$

$$\mathcal{U}_8 = \sum_{i=1}^{8} \cos\left((i-1)\tfrac{\pi}{16}\right)\tfrac{\pi}{16}$$

$$\mathcal{L}_{16} = \sum_{i=1}^{16} \cos\left(i\tfrac{\pi}{32}\right)\tfrac{\pi}{32} \approx 0.9501$$

$$\mathcal{U}_{16} = \sum_{i=1}^{16} \cos\left((i-1)\tfrac{\pi}{32}\right)\tfrac{\pi}{32}$$

$$\mathcal{L}_{32} = \sum_{i=1}^{32} \cos\left(i\tfrac{\pi}{64}\right)\tfrac{\pi}{64} \approx 0.9752$$

$$\mathcal{U}_{32} = \sum_{i=1}^{32} \cos\left((i-1)\tfrac{\pi}{64}\right)\tfrac{\pi}{64}$$

A Special Choice of $\hat{x}_i$

$$S_n = \sum_{i=1}^{n} \cos(\hat{x}_i)\,\Delta x$$

Since cos is the derivative of sin, the **mean-value theorem** says we can choose $\hat{x}_i$ so that

$$\cos(\hat{x}_i) = \frac{\sin x_i - \sin x_{i-1}}{\Delta x}$$

$$f'(\hat{x}_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

A Special Choice of $\hat{x}_i$

$$\mathcal{S}_n = \sum_{i=1}^{n} \cos(\hat{x}_i)\,\Delta x$$

$$= \sum_{i=1}^{n} \frac{\sin x_i - \sin x_{i-1}}{\pi/(2n)} \frac{\pi}{2n}$$

$$= \sum_{i=1}^{n} (\sin x_i - \sin x_{i-1})$$

$$= (\sin x_1 - \sin x_0) + (\sin x_2 - \sin x_1) + (\sin x_3 - \sin x_2) + \cdots + (\sin x_n - \sin x_{n-1})$$

$$= \sin x_n - \sin x_0$$

$$= \sin \tfrac{\pi}{2} - \sin 0 \;= 1 - 0 \;= 1$$

Since $\cos$ is the derivative of $\sin$, the **mean-value theorem** says we can choose $\hat{x}_i$ so that

$$\cos(\hat{x}_i) = \frac{\sin x_i - \sin x_{i-1}}{\Delta x} = \frac{\sin x_i - \sin x_{i-1}}{\pi/(2n)}$$

$\mathcal{L}_n \leq \mathcal{S}_n = 1 \leq \mathcal{U}_n$ for all n!

Therefore, the exact area is 1.

What the last example illustrates (a preview of things to come)

Suppose that $f(x) \geq 0$ for all x in $[a, b]$ and we want to find the area under its graph over $[a, b]$. If $f(x) = F'(x)$ for all x in $[a, b]$, then choosing $\hat{x}_i$ so that

$$f(\hat{x}_i) = \frac{F(x_i) - F(x_{i-1})}{\Delta x}$$

mean-value theorem

results in

$$\mathcal{S}_n = \sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1})\right) = F(x_n) - F(x_0) = F(b) - F(a).$$

So

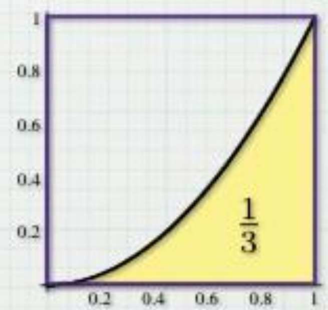
$$\mathcal{L}_n \leq F(b) - F(a) \leq \mathcal{U}_n \quad \text{for all } n.$$

Therefore, the *exact* area is $F(b) - F(a)$.

Example $f(x) = x^2$ on $[0, 1]$.

Since x^2 is the derivative of $\frac{1}{3}x^3$,

the area under the curve is $\frac{1}{3}1^3 - \frac{1}{3}0^3 = \frac{1}{3}$.



Is this not integration?